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New Approach to the Solution of Large, Full Matrix Equations

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A new approach to the solution of matrix equations resulting from integral equations is presented and applied to the solution of two-dimensional Neumann problems describing the inviscid, incompressible flow past an airfoil. The problem is reformulated in terms of a preselected set of mode functions giving an equivalent matrix equation to be solved for the mode-function expansion coefficients. Because of the inherent smoothness of the original problem, the coefficient problem can be solved approximately without significantly affecting the accuracy of the final solution. Very promising two-dimensional results are obtained and the extension of the method to three-dimensional problems is investigated. On the basis of these results it is shown that the computing time for the matrix solution for a large three-dimensional panel method calculation could be reduced by an order of magnitude compared with that required for a direct solution.

I. Introduction

THE result of the discretization of the governing integral equation for many physical problems is a large system of linear simultaneous equations. Such systems are usually highly "nonrandom" since the original physical problem generally possesses a smoothly varying behavior away from isolated singular regions. This is particularly true of the integral equations of many aerodynamic flow problems where the complexity comes from the geometry and a large number of points must be used to represent this adequately. This large number of points leads to a smoothly varying local behavior of the discrete quantities, but it also leads to the high computing costs associated with such calculations. The central thesis of the mode-function approach presented here is that, for problems of this kind, advantage can be taken of the locally smooth character by replacing the matrix equation with a related coefficient problem for certain preselected mode functions. Since a smooth function, in general, can be represented adequately in terms of a relatively small number of such functions, the size of the coefficient matrix can be significantly smaller than that of the original problem without seriously affecting the accuracy.

One problem of great practical significance is the calculation of the internal or external, inviscid, incompressible flow about a given shape. This is the classical boundary-value problem for Laplace's equation with a well-established mathematical pedigree (the Neumann problem) which leads to an integral equation of the second kind. The reduction of the large computation times associated with such three-dimensional "panel method" calculations is a primary aim of the mode-function method although it should be emphasized that the method is applicable to the solution of any matrix equation provided that the matrix possesses a dominant diagonal.

A study of the application of the mode-function method to two-dimensional Neumann problems has therefore been undertaken that demonstrates the validity of the concept. The extension of this work to three-dimensional problems is also discussed and the operation count presented in Sec. IV indicates that the mode-function method could offer significant savings in computer time compared to a direct matrix solution.

Before proceeding with an outline of the mode-function method it will be worthwhile to discuss the principal features of the panel methods that are used extensively for subsonic incompressible flow calculations. The problem is formulated as an integral equation to be solved for an unknown singularity distribution over the body about which the flow is to be computed.[1] For three-dimensional calculations the surface is represented by a large number of four-sided elements or "panels" which, in the version developed by Hess,[2] are on the actual body surface. A singularity distribution of a known form is then assumed on each panel whose strengths are determined by applying a normal velocity condition at a specified control point on each panel. In this basic method[2] the panels are assumed to be linear and the singularity distribution used is a constant strength source distribution over each panel. For lifting calculations an additional number of dipole distributions are introduced whose strengths are determined by Kutta conditions applied at points along the trailing edge.

For calculations involving complicated three-dimensional configurations the solution of the simultaneous equations is the dominant part of the computing effort and the method outlined here is therefore aimed at reducing this computing time. A more detailed discussion of the mode-function method is given in Ref. 3.

II. Mode-Function Approach

Having outlined the panel method to which the new approach is directed, a more detailed presentation of the mode-function theory can now be given. The fundamental assumption on which it is based is that both the singularity strength and the elements of the influence matrix must necessarily be slowly varying functions of position on the body surface. Therefore, their values on adjacent panels cannot be very different, so that solving the complete set of linear equations in which every unknown is assumed to be independent is wasteful in the sense that not all of the available information is being used. In the original problem, neighboring quantities are not in any sense independent. The mode-function method, therefore, seeks to fit the matrix and the singularity distribution to account for their continuity and dependency. The new problem becomes that of determining

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the "strength coefficients" of the chosen mode functions. Since, for example, a relatively small number of Fourier coefficients will contain as much information as a much larger number of data points, this coefficient problem can be of much lower order than the original problem.

Initially two kinds of mode functions were investigated based on Fourier series and on eigenvectors. The eigenvector method offers the advantage that it uses mode functions related to the particular problem, but the computational effort involved to calculate each eigenvector makes this approach impractical. The results presented here are, therefore, all based on the Fourier series approach which is attractive since the use of the fast Fourier transform (FFT) algorithm offers an extremely efficient way of calculating all of the Fourier coefficients for a given function.

In matrix notation, the source distribution for a nonlifting body, either two- or three-dimensional, is determined by an equation of the form

$$\rho - A\rho = f \tag{1}$$

where ρ and f are column vectors defining the unknown singularity distribution and the given "onset flow," respectively, while A is the matrix of influence coefficients. The precise form of this matrix will depend both on the type of the singularity distribution that is used and on the way in which the problem is discretized. The results presented here are all based on the case for which ρ is the source density, assumed to be constant over each element, and the elements are linear.

A vector $f = (f_1, f_2, \ldots f_m)$ can be regarded as a set of discrete values of a continuous function. By taking the independent variable to be the row or column number these values define the function at m equally spaced points and they can be used to calculate a Fourier series expansion for the function. The best rate of convergence for a finite-Fourier series is obtained by using a pure sine series, but since such an expansion will vanish at both ends of the interval an additional constant must be included in the series. Thus the series expansion for the f is given by

$$f_j = b_1 + \sum_{k=2}^{m} b_k \sin(j-1)(k-1)h \quad \text{where } h = \pi/m \tag{2}$$

In this expression $b_1 = f_1$ and the remaining coefficients are the standard Fourier sine coefficients of the function values $(f_j - b_1)$. Using the matrix notation this can be written

$$f = Gb \tag{3}$$

where G is the transformation matrix consisting primarily of Fourier sine functions. Similarly, the source density can be expressed as a sine expansion with coefficient $\boldsymbol{a}$,

$$\rho = Ga \tag{4}$$

Application of this process first to the rows of A and then to the columns of the resulting matrix leads to

$$A = GB\tilde{G} \tag{5}$$

where B is a matrix consisting primarily of Fourier coefficients, and $\tilde{G}$ the transpose of the matrix G.

The elements of b and B can be obtained by the application of the FFT algorithm to one- or two-dimensional arrays. Substitution of these expressions into Eq. (1) gives

$$Gb = Ga - GBDa \tag{6}$$

where $D = \tilde{G}G$ can be shown to have the form

$$D = \begin{bmatrix} m & \sigma_1 & \sigma_2 & \cdots & \sigma_{m-1} \\ \sigma_1 & m/2 & & & \\ \sigma_2 & & m/2 & & \bigcirc \\ \vdots & & \bigcirc & \ddots & \\ \sigma_{m-1} & & & & m/2 \end{bmatrix} \tag{7}$$

where

$$\sigma_k = \cot(kh/2) \quad (k \text{ odd})$$
$$= 0 \quad (k \text{ even})$$

The simple structure of the matrix D will make the calculation of the product BD very rapid.

Since the matrix G is nonsingular, Eq. (6) can be written as

$$b = a - BDa \tag{8}$$

This represents a set of m equations in m unknowns that is exactly equivalent to the original system of equations. However, if we assume that the known vector f and the solution vector ρ can be adequately represented in terms of their first n coefficients, then Eq. (8) can be truncated to give a system of n equations in n unknowns. The precise conditions under which this truncation is valid are discussed in detail in Sec. IIID at which time an improved scheme is developed. However, the following section will first present the results of the application of this simple truncation scheme to several airfoil calculations.

III. Application to Two-Dimensional Calculations

A. Basic Sine Series Fit

The initial airfoil used for testing the mode-function concept is a smooth airfoil, designated E33, defined by the "pole theory" of James.[4] This airfoil, shown in Fig. 1, has a region of high curvature around the trailing edge, but no actual surface slope discontinuity. The point numbering used begins at the rear stagnation point and proceeds in a counterclockwise direction around the leading edge and back to the starting point.

Before presenting the results obtained for this airfoil, it will be worthwhile to consider the fittability of the normal velocity matrix A of Eq. (1). Figure 2 shows the behavior of the 49th row of this matrix for the E33 airfoil defined by 96 panels. In this figure A_{ij} represents the negative of the normal velocity induced at a control point near the leading edge by unit source distributions on each of the panels. Using the basic Neumann matrix based on linear panels with constant source strength gives rise to a zero diagonal term which in turn leads to the spike shown in the figure. This effect can be reduced by using the higher-order matrix formulation[5] that takes into account the local airfoil curvature. However, numerical results indicate that this is of little importance and so the results presented here are all based on the lower-order formulation. Of greater significance to the matrix fittability is the secondary peak caused by source panels on an adjacent surface. Figure 3 shows the behavior of A_{ij} for row 21 illustrating the influence of each of the source panels on a control point on the upper surface. As the airfoil becomes thinner this effect becomes more severe and it can therefore be expected to govern the number of terms required to achieve an accurate solution.

Figure 4 shows the velocity distribution plotted against airfoil arc length for this smooth airfoil. Both the exact solution, obtained by direct matrix solution of Eq. (1), and the truncated solution obtained by using 32 terms in Eq. (8)

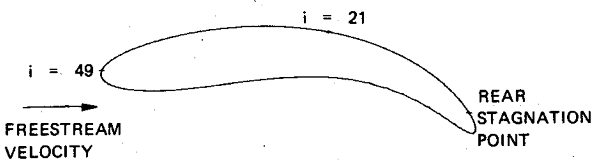

Fig. 1 The E33 96-point airfoil.

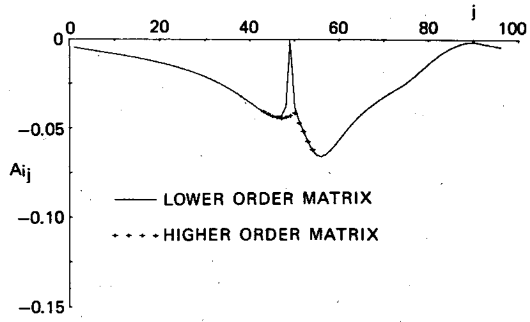

Fig. 2 The behavior of A_{ij} matrix for $i=49$ on the 96-point E33 airfoil.

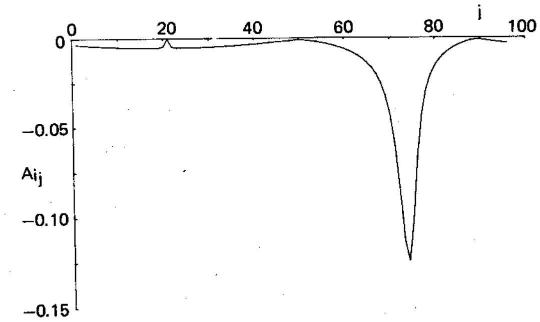

Fig. 3 The behavior of A_{ij} matrix for $i=21$ on the 96-point E33 airfoil.

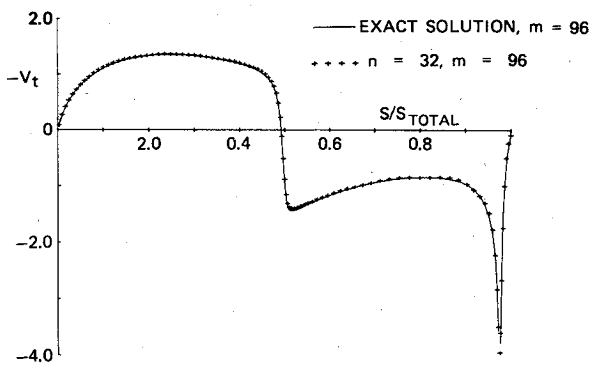

Fig. 4 Velocity distribution for E33 airfoil; basic Fourier method.

are shown. This truncated solution is very close to the exact solution even through the trailing-edge region where there is a sharp peak in the velocity. However, as the number of terms is reduced, the truncated solution becomes unacceptably oscillatory.

This basic method was applied to a 20%-thick airfoil with a sharp trailing edge, shown in Fig. 5 and designated BCC2. The results of Fig. 6 show that an excellent approximation can be obtained using only 12 terms in the truncated solution for the zero lift case. However, the results become unsatisfactory for this same airfoil at 10 deg. A sharp oscillation in the velocity distribution develops near the trailing edge, and this is present even when as many as 40 terms are used. An examination of the way in which the Kutta condition is

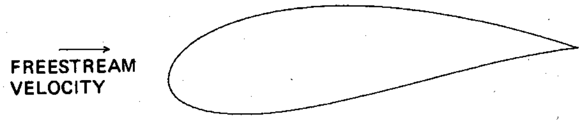

Fig. 5 Nonlifting BCC2 airfoil.

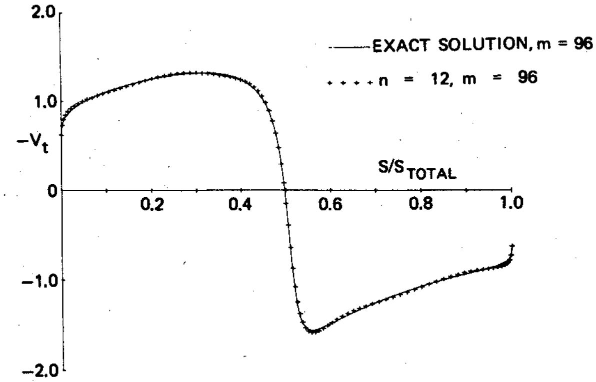

Fig. 6 Velocity distribution for nonlifting BCC2 airfoil; basic Fourier method.

satisfied for a lifting airfoil reveals the cause of this failure. The singularity distribution consists of two components; the first, a pure source distribution that satisfies the boundary condition on the airfoil due to the onset flow, and the second, a constant vortex plus a varying source distribution that together satisfy a zero normal velocity on the airfoil. These components are then combined to satisfy the Kutta condition at the trailing edge. For a lifting airfoil, each of these components becomes singular at a sharp trailing edge, but in the full Neumann solution these two singular terms cancel and the resulting source distribution is finite at the trailing edge. However, when the solution is truncated, these two singular components are approximated inaccurately and the errors are not eliminated when the combined solution is formed. In order to overcome this problem, two modifications to the basic method are presented, the first aimed at improving the fitting method in order to cope with singular functions while the second reformulates the Kutta condition so that it is satisfied simultaneously with the other equations thereby ensuring that the source distribution to be fitted is finite at the trailing edge.

B. Linear Fit

It will be recalled that the basic sine series fit, Eq. (2), assumes that the function to be fitted is periodic with $f_1 = f_{m+1}$. This is true in theory for a closed body, but if the function to be fitted contains a singularity at the trailing edge, there will be a large discontinuity between the values f_m and f_{m+1}. However, by subtracting a linear function of the point number, it is possible to ensure that the values to be fitted are zero at both ends of the interval. Therefore, to fit the vector $f = (f_1, f_2, \ldots f_m)$ we first define the vector g by

$$g_j = f_j - \frac{m-j}{m-1} f_1 - \frac{j-1}{m-1} f_m \quad j=1,2,\ldots m \tag{9}$$

Thus, $g_1 = g_m = 0$ and this vector can be expressed as a sine series, and Eq. (2) will be replaced by

$$f_j = \frac{m-j}{m-1} b_1 + \frac{j-1}{m-1} b_2 + \sum_{k=3}^{m} b_k \sin(k-2)(j-1)h \tag{10}$$

where $b_1 = f_1$, $b_2 = f_m$ and where h is given by $h = \pi/(m-1)$. Thus the transformation matrix G in Eq. (3) will still consist primarily of Fourier sine functions. This modified fitting procedure is applied both to the influence matrix and the column vectors so that the details of the application closely follow those given in Sec. II for the basic method. The matrix product $D=\tilde{G}G$ can be shown to take on the relatively simple structure

$$D=\begin{bmatrix} c_1 & c_2 & \sigma_1 & \sigma_2 & \cdots & \sigma_{m-2} \\ c_2 & c_1 & \sigma_1 & -\sigma_2 & \cdots & (-1)^{m-1}\sigma_{m-2} \\ \sigma_1 & \sigma_1 & (m-1)/2 & & & \\ \sigma_2 & -\sigma_2 & & (m-1)/2 & \bigcirc & \\ \vdots & \vdots & \bigcirc & & \ddots & \\ \sigma_{m-2} & (-1)^{m-1}\sigma_{m-2} & & & & (m-1)/2 \end{bmatrix} \tag{11}$$

where

$$c_1=m(2m-1)/[6(m-1)] \qquad c_2=m(m-2)/[6(m-1)]$$

and

$$\sigma_i=\tfrac{1}{2}\cot(\tfrac{1}{2}ih), \qquad i=1,2,\ldots m$$

This can now be substituted into Eq. (8) to give a new coefficient equation that is solved in precisely the same way as in the basic method.

Figure 7 shows the truncated velocity distribution obtained by using this linear fit on the BCC2 airfoil at 10 deg. Using 32 terms in the truncated series gives reasonable agreement with the exact solution although as the number of terms used is reduced this solution again becomes oscillatory. This indicates that the singularity occurring in the source distribution at the trailing edge will still cause problems when an approximate matrix solution is used.

C. Combined Kutta Condition

The previous section illustrates that some improvement can be achieved by accounting for the singular behavior of the source strength near the trailing edge. However, in order to reduce the number of terms required for an adequate solution, further investigation of this singular behavior is required. On the other hand, this section considers the possibility of reformulating the problem so that the calculated source density is well behaved. It was seen in Sec. IIIA that the basic sine fit gave excellent results for a nonlifting airfoil for which the basic source distribution automatically satisfies the Kutta condition. Therefore, if the circulation is treated as an additional unknown, the Kutta condition can be satisfied simultaneously with the source strength equations and the resulting source distribution will be finite at the trailing edge.

This will now give a system of $m+1$ equations in $m+1$ unknowns that could, in principle, be solved by the direct application of the fitting procedures described earlier. However, this is not practical since the additional values added to the matrix and the column vectors will destroy their continuous nature upon which this method relies. For instance, a new column vector would be defined consisting of the source strengths and the circulation. Since the circulation will not be related directly to the adjacent source values, there will be a discontinuity in this modified vector. Such discontinuities cannot be represented adequately by continuous functions and so the total circulation will be approximated poorly, and the resulting solution will be slowly convergent in n.

It is, however, possible to eliminate the circulation numerically and then apply the fitting technique to the modified $m\times m$ matrix. This approach has been applied to the lifting BCC2 airfoil and a typical result is shown in Fig. 8. This figure is based on a 49 point airfoil definition from which it can be seen that a 16 term expansion provides a good solution. In addition to using the combined Kutta condition, this result uses the linear fit of Sec. IIIB together with a technique referred to as "σ-smoothing." Under this approach the calculated Fourier coefficients are each multiplied by an attenuation factor that is unity for the first coefficient decreasing progressively to zero for the first coefficient to be neglected. This has the effect of reducing some of the higher frequency oscillation that occurs in the solution although it should be noted that the overall error will be increased.

D. Improved Matrix Truncation Scheme

While the results presented in the previous section are encouraging, they are based on the BCC2 airfoil which is a very thick airfoil. To provide a more realistic test case, a thin lifting airfoil was created by scaling and rotating the coordinates for the BCC2 airfoil to give an 8%-thick airfoil at an incidence of approximately 5 deg shown in Fig. 9. As the

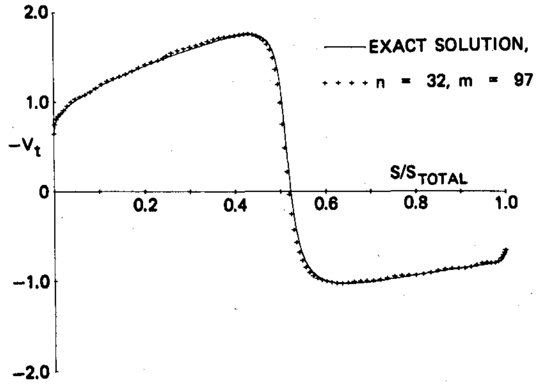

Fig. 7 Velocity distribution for lifting BCC2 airfoil, $\alpha \simeq 10$ deg; Fourier method with linear fit.

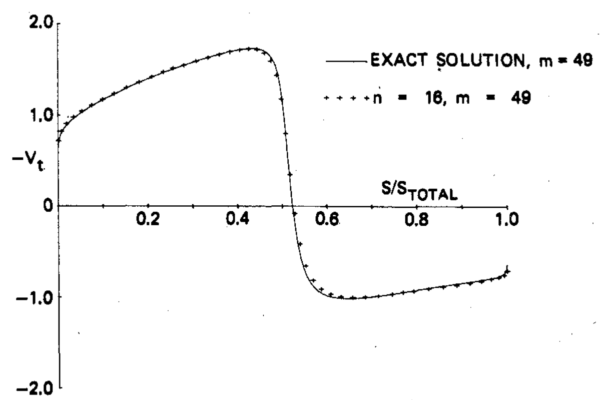

Fig. 8 Velocity distribution for lifting BCC2 airfoil, $\alpha \simeq 10$ deg; Fourier method with combined Kutta condition, linear fit, and σ smoothing.

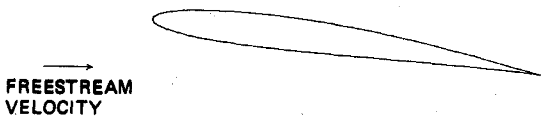

Fig. 9 Thin lifting airfoil, $\alpha \simeq 5$ deg.

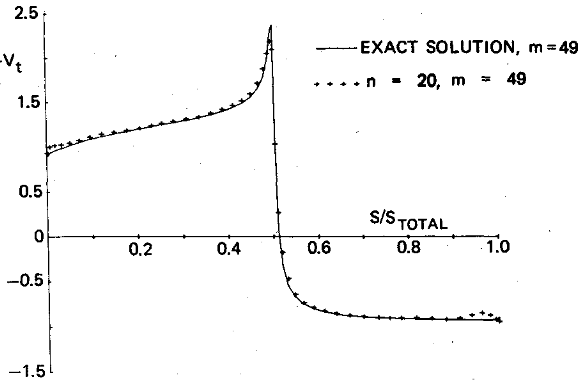

Fig. 10 Velocity distribution for thin lifting airfoil, $\alpha \approx 5$ deg; Fourier method with combined Kutta condition, linear fit, and σ smoothing.

thickness is reduced the peak in the influence matrix, resulting from the effects of source elements on the opposite surface and illustrated in Fig. 3, becomes more pronounced. This increased peak impairs the fittability of the matrix leading to the effect seen in Fig. 10 in which the method of the previous section has been applied to this thin airfoil. For a 20 term expansion, for which $n/m \sim 2/5$, the results are inadequate near the trailing edge, while the use of a greater number of terms would detract from the computational efficiency of the mode-function method. With the use of the FFT algorithm, all of the expansion coefficients for the transformed matrix Eq. (8) are known, although the size of the truncated matrix equation to be solved must be kept as small as possible in order to reduce the computation time. This truncation process therefore has been examined with the aim of using all of the available information to obtain an improved solution while explicitly solving only a small fraction of the equations.

The transformed coefficient equations to be solved are given by Eq. (8) where D is given either by Eq. (7) for the basic fit, or by Eq. (11) for the linear fit. In either case the matrix B will consist primarily of Fourier coefficients. Provided that the matrix A is sufficiently smooth, the coefficients of the higher harmonics will be small compared with those of the lower harmonics. Thus, in general, the most significant information contained by the matrix B will be concentrated into the first few rows of the first few columns, i.e., the magnitude of the entries of B will decrease, in general, as the row or column number increases. Consideration of the structure of the matrix D leads to the conclusion that the matrix product BD will have the property that, in general, its elements will decrease in magnitude as the row number increases.

In order to express these ideas in mathematical terms, the matrix product BD and the column vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in Eq. (8) can be partitioned to separate the first n rows and columns where $n < m$. Thus

$$BD = C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{12}$$

$$\boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix} \tag{13}$$

where C_{11} is an $n \times n$ matrix, while $\boldsymbol{a}_1$ and $\boldsymbol{b}_1$ each have n entries. From the preceding discussion we can assert that, provided n is suitably chosen, the largest element in the $(m-n) \times n$ matrix C_{21} will be small compared with the largest element in C_{11}. Similarly, the largest element in the $(m-n) \times (m-n)$ matrix C_{22} also will be small.

Substitution of Eqs. (12) and (13) into Eq. (8) leads to the two simultaneous matrix equations

$$(I - C_{11})\boldsymbol{a}_1 - C_{12}\boldsymbol{a}_2 = \boldsymbol{b}_1 \quad -C_{21}\boldsymbol{a}_1 + (I - C_{22})\boldsymbol{a}_2 = \boldsymbol{b}_2 \tag{14}$$

where the symbol I has been used to represent the $n \times n$ unit matrix in the first equation and the $(m-n) \times (m-n)$ unit matrix in the second equation. Now if n is "sufficiently large" and $\boldsymbol{b}$ and C "sufficiently smooth," then $\boldsymbol{b}_2$ and C_{21} can be neglected and these two equations can be approximated by

$$(I - C_{11})\boldsymbol{a}_1 = \boldsymbol{b}_1 \qquad \boldsymbol{a}_2 = 0 \tag{15}$$

which is the truncated form of the equation that has been used for all the results presented so far. At this stage the significance of the dominant diagonal becomes clear. If the unit matrices were absent from Eq. (14) then the terms C_{21}, C_{22}, and $\boldsymbol{b}_2$ would all have the same order of magnitude, and the approximation $\boldsymbol{a}_2 = 0$ does not necessarily follow.

Equation (14) is exactly equivalent to the original matrix equation, but since C_{22} is small, it follows that we can make the approximation

$$(I - C_{22})^{-1} = I + C_{22} + \ldots \tag{16}$$

and so $\boldsymbol{a}_2$ can be eliminated to give

$$[I - C_{12}(I + C_{22})C_{21}(I - C_{11})^{-1}](I - C_{11})\boldsymbol{a}_1 = \boldsymbol{b}_1 + C_{12}(I + C_{22})\boldsymbol{b}_2 \tag{17}$$

Again, since the matrix C_{21} also is small, we can introduce the second approximation

$$[I - C_{12}(I + C_{22})C_{21}(I - C_{11})^{-1}]^{-1} = I + C_{12}(I + C_{22})C_{21}(I - C_{11})^{-1} + \ldots \tag{18}$$

Therefore, Eq. (14) can be approximated by

$$(I - C_{11})\boldsymbol{a}_1 = [I + C_{12}(I + C_{22})C_{21}(I - C_{11})^{-1}] \times [\boldsymbol{b}_1 + C_{12}(I + C_{22})\boldsymbol{b}_2]$$

and

$$\boldsymbol{a}_2 = (I + C_{22})(\boldsymbol{b}_2 + C_{21}\boldsymbol{a}_1) \tag{19}$$

The explicit calculation of any of the matrix products involved in the right-hand side of Eq. (19) is out of the question. However, the terms on the right-hand side can each be calculated by performing a succession of products between matrices and column vectors, together with one matrix inversion required to find $(I - C_{11})^{-1}$. It can be shown therefore that a full solution to Eq. (19) requires $0(n^3/3)$ operations to factorize the matrix $(I - C_{11})$ together with $0(3m^2)$ operations required to perform all of the products necessary to solve for $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Thus for an additional $0(3m^2)$ operations, the solution of Eq. (19) gives an approximation to all of the expansion coefficients and takes into account all of the information available from the FFT fitting procedure.

The scheme defined by Eq. (19), with the linear fitting of Sec. IIIB used to determine the expansion coefficients, has been applied to the thin lifting airfoil and excellent results were obtained. Figures 11 and 12 show the approximate solutions obtained for $m = 49$ with $n = 8$ and $n = 12$, respectively. Thus for $n/m \sim 1/4$ the agreement is very good while

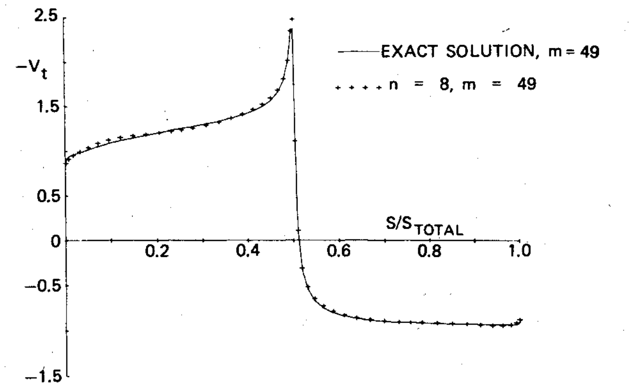

Fig. 11 Velocity distribution for thin lifting airfoil, $\alpha \simeq 5$ deg; Fourier method with improved matrix truncation scheme and linear fit.

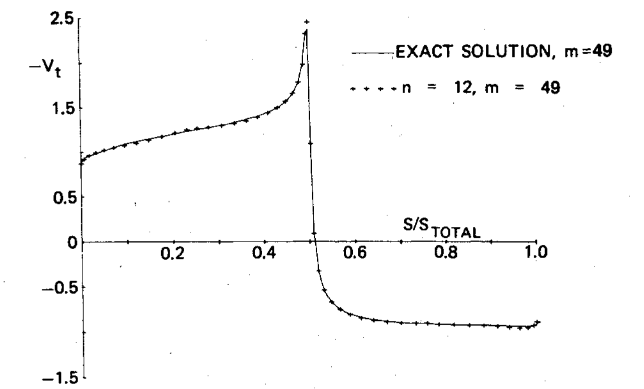

Fig. 12 Velocity distribution for thin lifting airfoil, $\alpha \simeq 5$ deg; Fourier method with improved matrix truncation scheme and linear fit.

even for $n/m \sim 1/6$ the solution may be adequate for many purposes.

These results serve to demonstrate the potential for this particular formulation of the mode-function method. It should be pointed out, however, that the only two approximations involved in the method are given by Eqs. (16) and (18). Either of these equations could be made more accurate by the inclusion of additional terms at the expense of increasing the complexity of the calculation. A careful balance therefore must be maintained between the need to minimize the ratio n/m and the additional number of arithmetic operations involved.

IV. Application to Three-Dimensional Problems

A. Basic Approach

The results presented so far are all concerned with two-dimensional applications although, as stated in Sec. II, the numerical formulation of a three-dimensional Neumann problem is related closely to the two-dimensional case. In general, a three-dimensional application will involve a larger number of panels, and the structure of the influence matrix and the singularity vector will be rather different. This structure therefore will be examined before considering a three-dimensional application of the mode-function method.

For a given three-dimensional configuration, in order to formulate the corresponding matrix equation for the singularity strengths, a numbering system for the panels has to be selected. A natural choice is to consider successive streamwise sections so that a wing, for instance, would be composed of k strips each consisting of m panels. The corresponding source distribution vector would, therefore, consist of k blocks, each containing m values associated with the successive streamwise sections. This vector is approximately periodic in behavior since the values around one strip would be related closely to the corresponding values on an adjacent strip. In addition, there will be discontinuities in this vector associated with the jump across the trailing edge from one strip to the next. In the light of the two-dimensional results, a global fit of this function will not produce a very rapidly convergent mode-function expansion and a sectional fitting procedure appears to be preferable.

When applied to the influence matrix, the fitting procedure first fits the rows and then the columns of the resulting matrix. Provided that the matrix can be held in the core store of the computer, this process can be accomplished efficiently. However, for a three-dimensional problem, this matrix is held on disk storage and the penalties involved in accessing this matrix by rows and then by columns could be considerable. This also indicates the need to fit the matrix in smaller blocks each of which could be held in core and again the use of $m \times m$ blocks corresponding to the individual streamwise sections appears to be the natural choice.

One final argument against the use of a global fitting method concerns the FFT algorithm. The results of Sec. IIID clearly demonstrate the need to obtain all of the expansion coefficients for both the matrix and the right-hand side, and this in turn implies that the FFT algorithm must be used. This method is able to derive all of the Fourier coefficients for an $m \times m$ matrix in $0(2S_m m^2)$ operations where S_m is the sum of all of the prime factors of m. Therefore, it is necessary to choose m carefully to avoid including large prime factors that could excessively increase S_m. For instance, if m itself were a prime number, then $S_m = m$ and clearly the fitting procedure would be an $0(2m^3)$ operation and this would be uneconomic compared with a direct solution. If the fitting procedure is applied to two-dimensional strips for which typically $m < 100$ this limitation is not unduly restrictive. However, as the value of m is increased, the number of acceptable values becomes more sparse, and the use of a global fitting approach will be too restrictive.

The preferred approach to be adopted for a three-dimensional problem, therefore, would be to subdivide the matrix into blocks corresponding to the sectional strips. The diagonal blocks will be related very closely to the matrices for the two-dimensional problems considered in this report while the off-diagonal blocks will define the influence of all the points within one sectional strip upon all the control points of another sectional strip. It would not be necessary for these individual blocks to have the same dimensions, the only restriction being that each of these individual dimensions must be an acceptable value for the FFT algorithm. With the unknown source density vector and the given right-hand side vector similarly subdivided, the analysis follows very closely that given for the two-dimensional method presented here. In particular, the improved scheme presented in Sec. IIID could be used and so the time saving to be expected from the mode-function method can be based on this two-dimensional study. The arithmetic operation count for such a three-dimensional method therefore will be considered in more detail.

B. Operation Count for Three-Dimensional Problem

Let the body be divided into k strips and assume for the sake of simplicity that each strip consists of m panels. The calculation of all of the Fourier coefficients B of an $m \times m$ matrix involves $0(2 \cdot S_m m^2)$ operations where S_m is the sum of all of the prime factors of m. Typically, for m between 50 and 100, S_m will be between about 11 and 15. The number of operations required to set up the matrix product BD of Eq. (8) will be $0(2m^2)$. Thus, for the complete matrix, $0[2k^2(S_m+1)m^2]$ operations will be required to calculate all of the coefficients required for Eq. (8). Following Sec. IIID, if kn of these equations are solved explictly, $0(k^3n^3/3)$ operations are required to perform the triangular decomposition and a further $0(3k^2m^2)$ operations will be required to complete the solution of Eq. (19). On the other hand, $0(k^3 m^3/3)$ operations are required for a direct solution of the original matrix equation. Thus a guide to the reduction in computing time that could be achieved through the use of the mode-function method is given by the ratio of these two operation counts, i.e.,

$$\left(\frac{n}{m}\right)^3 + \frac{3(2S_m+5)}{km} \tag{20}$$

The results of Sec. IIID indicate that it is realistic to assume that $n/m = 1/3$ although in practice it may be possible to use a smaller value for the ratio. For a typical three-dimensional configuration with $k = 40$ and $m = 50$, $S_m = 12$ and the value of Eq. (20) becomes approximately 1/12. Therefore, for this particular case, the mode-function method would be about 12 times faster than the direct solution. The computing effort in this case would be roughly equally divided between the fitting of the matrix and the solution of the resulting equations. As the number of panels is further increased, the time required for the fitting beomes a smaller proportion of the total time. For such cases it therefore would be worthwhile to improve the approximations involved in Sec. IIID in order to achieve lower values for n/m. Thus for a 4000 panel case with the assumption that $n/m = 1/4$ the mode-function method would be 26 times faster than the direct solution, and an 8000 panel case would be 38 times faster. On the other hand, an iterative solution would require $0(Ik^2m^2)$ operations for one flow condition where I is the number of iterations required. For a simple configuration such methods require 10-15 iterations, although for more complicated configurations, such as those involving internal flow, more iterations may be required. Thus for one flow calculation the iterative method could, at best, be 4-5 times faster than the mode-function method for the examples considered above. However, the mode-function method does possess several properties of the direct solution that are not present in the iterative method. For instance, both the calculation of flows at different angles of attack and the simulation of boundary-layer effects require at least two flow solutions. Each such solution involves a complete calculation for the iterative method whereas for both the direct and the mode-function methods the additional solutions can be computed for very little extra cost. In addition, the solution time for the iterative method is dependent on the number of iterations required. As this can vary significantly for different configurations, this method is not always predictable. However, provided that n/m is specified beforehand, the solution time for the mode-function method will be totally predictable. Thus, any apparent advantage of the iterative method could very quickly be lost in practice.

V. Conclusions

The mode-function concept provides a new method for obtaining approximate solutions to large systems of linear simultaneous equations. The quantities defining many such systems can be considered as discrete values of smoothly varying functions and so they can be expressed in terms of a given set of continuous mode functions. In this way the original problem can be reformulated to give a new set of equations whose solution gives the mode-function expansion coefficients for the solution to the original problem. By truncating these expansions the number of equations to be solved is reduced and an approximate solution to the original set of equations is obtained.

The mode-function approach has been applied to a two-dimensional panel method calculation of the inviscid, incompressible flow about an airfoil. The results presented here are all based on the use of Fourier series expansions using the fast Fourier transform algorithm to provide an extremely efficient method for calculating the expansion coefficients for the influence matrix. In addition, an approach to the more time consuming three-dimensional flow calculations is proposed. On the basis of the two-dimensional results obtained, a substantial reduction in computer time relative to the direct solution, could be achieved.

In its basic form, the mode-function expansion in terms of a truncated sine series does not give a satisfactory method for approximating the flow over a realistic airfoil. An accurate solution would require the inclusion of too many terms to permit a significant reduction in the computing time. However, several methods of improving this basic approximation have been examined of which the most promising is the improved matrix truncation scheme presented in Sec. IIID. With this approach an approximate solution is calculated for all of the Fourier coefficients for the unknown source density. Accurate results for a thin lifting airfoil are obtained while solving explicitly only about 1/4 of the original number of equations.

For three-dimensional problems the influence coefficient matrix would need to be fitted by blocks, each block corresponding to a streamwise strip of panels. When applied in this manner the operation count presented in Sec. IV suggests that the mode-function method could be about 12 times faster than a direct solution for a 2000 panel configuration. The mode-function method therefore offers a promising approach to the reduction of the computing costs for large three-dimensional Neumann calculations. Alternatively, the method would offer the capability of handling larger numbers of unknowns for the same computational cost. In addition, the method should be applicable to the solution of any large system of linear simultaneous equations provided that the matrix possesses a dominant diagonal and that it is otherwise smooth.

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